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Autonomous under University of Mysore
Re-accredited by NAAC with 'A' Grade, CGPA: 3.21
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DEPARTMENT OF MATHEMATICS

Chapter 1

Rings

1.1 Rings and homomorphisms

Let $\mathcal{R}$ be a ring. We have an object ($\mathcal{R}$,+,-,$\cdot$). This object contains a set with 3 operations. The operations are maps from $\mathcal{R}$x$\mathcal{R} \to \mathcal{R}$. The subset ($\mathcal{R}$,+) is a commutative group, and the subset ($\mathcal{R}$,$\cdot$) is a semi-group or monoid. In the ring there is an element 0 such that 0x = x0 = 0 for all x in $\mathcal{R}$. In the ring there is also an identity element 1, with the property that 1x = x1 = x for all x in $\mathcal{R}$. The ring $\mathcal{R}$ also satisfies a distribution law with respect to the operations of addition and subtraction.

Examples of rings:

1. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, C (note $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields);
2. $\mathbb{Z}[x]$, $\mathbb{Q}[x]$, $\mathbb{R}[x]$, $\mathbb{C}[x]$

Definition 1.1.1. If $\mathcal{R}$ is a commutative ring, then $\mathcal{R}[x]$ is the ring of polynomials over $\mathcal{R}$:

$$\mathcal{R}[x] = \{ \sum_{i=1}^{n} a_i x^i \mid a_i \in \mathcal{R}, \text{ with operations } +, -, \text{ and } \cdot \ \}$$

Definition 1.1.2. Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two rings. A homomorphism from $\mathcal{R}_1$ to $\mathcal{R}_2$ is a map

$$\phi : \mathcal{R}_1 \to \mathcal{R}_2$$

which preserves the operations:

$$\phi(x + y) = \phi(x) + \phi(y), \qquad \phi(xy) = \phi(x)\phi(y)$$

Definition 1.1.3. The kernel of a homomorphism $\phi : \mathcal{R}_1 \to \mathcal{R}_2$ is defined to be the preimage of 0 and its denoted by Ker(ϕ).

$$\text{Ker}(\phi) = \phi^{-1}(0) = \{ x \in \mathcal{R} \mid \phi(x) = 0 \}$$

Note that Ker(ϕ) measures the injectivity of ϕ.

Lemma 1.1.4. *The homomorphism ϕ is injective if and only if* $\ker\phi = 0$.

Proof. The only if part is obvious. If $\phi(x_1) = \phi(x_2)$ then $\phi(x_1 - x_2) = 0$ so $x_1 - x_2 = 0$. □

The homomorphism ϕ can be factorized as a compoisition of surjective map and an injective map

$$\mathcal{R}_1 \xrightarrow[\text{onto}]{\phi} \phi(\mathcal{R}_1) \hookrightarrow \mathcal{R}_2$$

The image $\phi(\mathcal{R}_1)$ is a *subring* of $\mathcal{R}_2$, while it is a *quotient ring* of $\mathcal{R}_1$.

Some notations

- Surjective map: $\twoheadrightarrow$
- injective map: $\hookrightarrow$ or $\rightarrowtail$

1.2 Ideals and quotients

Now we want to study the structure of $\ker\phi$. Let $\ker\phi = I$.
Property 1 *If $x \in I$ and $y \in I$ then $x + y \in I$.* Thus I is an Abelian subgroup of $\mathcal{R}$ under addition.
Property 2 *If $x \in \mathcal{R}$ and $y \in I$ then $xy \in I$.* Indeed

$$\phi(xy) = \phi(x)\phi(y) = 0.$$

Note that if $1 \in I$, then $\phi(1) = 0$ which implies

$$\phi(x) = \phi(x1) = 0$$

for all $x \in \mathcal{R}$. Thus for the most part, we assume
Property 3 *1 is not in I*

Definition 1.2.1. Let I $\hookrightarrow \mathcal{R}$ be a subset. We say I is an ideal if I satisfies two properties:

1. $x \in I, y \in I$ implies $x + y \in I$
2. $x \in I$, $y \in \mathcal{R}$ implies $xy \in I$

Theorem 1.2.2. *A subset I of $\mathcal{R}$ is the kernel of a homomorphism ϕ :$\mathcal{R} \to \mathcal{R}'$, if and only if I is an ideal.*

Here is some machinery to start: Let $\mathcal{R}'$ denote the quotient $R/\sim$ of R modulo the relation $\sim$: where

$$x_1 \sim x_2 \quad \text{if and only if} \quad x_1 - x_2 \in I.$$

Step 1. Show this is an equivalence: indeed,

$$x_1 - x_2 \in I, \quad x_2 - x_3 \in I, \quad \text{then} \quad x_1 - x_3 \in I.$$

Notation: let $x \in \mathcal{R}$. The class of x in $\mathcal{R}'$ is denoted by $x + I$ or $x \pmod{I}$.
Step 2. Define addition and multiplication on $\mathcal{R}'$:

$$(x_1 \pmod{I}) + (x_2 \pmod{I}) = (x_1 + x_2) \pmod{I}$$

$$(x_1 \pmod{I})(x_2 \pmod{I}) = x_1 x_2 \pmod{I}$$

Step 3. Show $\mathcal{R}'$ is a ring.
Step 4. Define a map

$$\phi : \mathcal{R} \to \mathcal{R}', \quad \phi(x) = x \pmod{I}.$$

Show ϕ is a homomorphism and $\ker\phi = I$.

1.3 Special ideals and rings

We want to introduce some special ideals and rings through study of examples: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$[x], $\mathbb{Q}$[x], $\mathbb{R}$[x], $\mathbb{C}$[x].

Fields

Let $I \hookrightarrow \mathbb{Q}$ be an ideal. If I is non-zero, then there is some non-zero element $a \in I$. If $a \in I$, then $a^{-1} \in I$, so $aa^{-1} = 1 \in I$. Thus the only ideal in $\mathbb{Q}$ not containing 1, is the zero ideal. This leads us to the following definition:

Definition 1.3.1. A *Field* is a ring whose only ideal is the zero ideal.

Another definition could be:

Definition 1.3.2. A *Field* is a ring in which every non-zero element is invertible. That is for all $x \in \mathcal{R}$ there exists $y \in \mathcal{R}$ such that $xy = 1$.

The equivalence of these two definitions is easy to see. If there were some non-zero element $x \in \mathcal{R}$ that was not invertible, then (x) will be non-zero and $(x) \neq \mathcal{R}$. If every non-zero element of $\mathcal{R}$ is invertible then clearly the only ideal of $\mathcal{R}$ is the zero ideal.

It is clear that $\mathbb{R}$ and $\mathbb{C}$ are also fields.